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Diagonalisation of a colouring problem (on a strip)

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Abstract. A bond colouring problem on a honeycomb lattice, equivalent to evaluating the residual entropy of a four-state antiferromagnetic Potts model, is considered. The transfer matrix for strips with three or four hexagons in each row and any number of rows is diagnonalised.

1. Introduction

Problems which may be solved exactly in two dimensions in statistical mechanics are very interesting not only because they involve various mathematical techniques but also for their possible relations with bidimensional quantum field theories [1]. In recent years, bidimensional ferromagnetic models in finite or infinite domains, with a variety of boundary conditions at their critical temperature, were studied and their quite interesting relations with conformally invariant field theories [2] were exhibited.

Bidimensional antiferromagnetic models seem less well understood and their possible relations with a corresponding field theory in a continuous space remain problematic [3]. In this paper some investigations are described concerning the residual entropy of a combinatorial model, which was solved by Baxter in a bidimensional infinite domain long ago [4]. The model was defined on a regular honeycomb lattice with toroidal boundary conditions. In the present paper, the model is studied for strips of finite width and infinite length, with the same boundary conditions.

In § 2 we outline the construction of a set of projectors which should be useful for exactly diagonalising a variety of transfer matrices defined on honeycomb lattices with cylindrical boundary conditions.

In § 3 we report the evaluation of the finite transfer matrix for the model studied here and its exact diagonalisation. The leading eigenvalue, computed by Baxter for the infinite domain in both directions, is quoted to estimate the rate of convergence of the finite-width-strip computation.

The combinatorial model of Baxter may be regarded as an antiferromagnetic Potts model at zero temperature and we hope that the present study will be useful for understanding general properties of the ground state of antiferromagnetic models. The combinatorial model is also related to a general class of colouring problems of planar graphs. These relations will be discussed in a paper in preparation.

2. The group algebra, the invariant basis elements and projectors

Let us consider a bidimensional square lattice with n sites in each row, a random variable x_i associated with site i in a given row, with cylindrical boundary conditions,

$x_{n+i}=x_i$, $i=1,\ldots,n$, and any number of horizontal rows. It is then natural to consider the transfer matrix associated with a horizontal row and its invariance properties. In the usual way $T_{x_1,\ldots,x_n;y_1,\ldots,y_n}$ relates the configurations $\bar{x}$ and $\bar{y}$ on two consecutive rows and the product of transfer matrices corresponding to m consecutive rows defines a partition function for a lattice with $m+1$ rows and n columns.

We are interested in the exact diagonalisation of the one-row transfer matrix (for some value of n).

One expects the transfer matrix to be invariant under the transformation

$$T'_{x_2,\ldots,x_n,x_1;y_2,\ldots,y_n,y_1}=\tau T\tau^{-1}=T_{x_1,x_2,\ldots,x_n;y_1,\ldots,y_n} \tag{2.1}$$

where τ is the permutation

$$\begin{pmatrix} 1 & 2 & n-1 & n \\ 2 & 3 & n & 1 \end{pmatrix}$$

since this would correspond to a coordinate system translated one lattice space to the right with respect to the previous one.

Furthermore, in a problem with left/right invariance, one expects the transfer matrix to be invariant also for the transformation

$$T'=\sigma T\sigma'=T \tag{2.2}$$

where

$$\sigma=\begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ n & n-1 & \ldots & 2 & 1 \end{pmatrix}. \tag{2.3}$$

τ and σ are generators of the dihedral group D_n, of order $2n$ and the transfer matrix should be invariant under the transformation

$$T'=dTd^{-1}=T \qquad d\in\mathrm{D}_n. \tag{2.4}$$

We consider the permutation group S_n. For each $\pi_R\in\mathrm{S}_n$ we consider its conjugate elements by a generic element $d\in\mathrm{D}_n$:

$$\{\pi_R\}=\text{set of } d\pi_R d^{-1} \qquad d\in\mathrm{D}_n. \tag{2.5}$$

The classes of conjugate elements are a partition of S_n. For the lattice with three sites in each row, the permutation group S_3 coincides with the dihedral group D_3, then the partition in conjugate classes is just the partition of permutations in the three classes with different cycle lengths

$$1:\ (1)(2)(3)=e \qquad 2:\ (12),(13),(23) \qquad 3:\ (123),(132). \tag{2.6}$$

For the lattice with four sites in each row one easily obtains the eight classes:

$$\begin{aligned} &1:\ (1)(2)(3)(4)=e\\ &2:\ (12),(14),(23),(34)\\ &3:\ (13),(24)\\ &4:\ (123),(132),(124),(142),(134),(143),(234),(243)\\ &5:\ (12)(34),(14)(23) \end{aligned} \tag{2.7}$$

6: (13)(24)

7: (1234),(1432)

8: (1342),(1243),(1324),(1423).

Next we consider the group algebra [5] A_n of elements $x = \Sigma_R x_R \pi_R$ where $\pi_R \in S_n$ are regarded as the basis of a linear vector space of dimension $n!$ and the coefficients x_R are rational numbers. We are particularly interested in the elements:

$$\alpha_1 = e \qquad \alpha_2 = (12)+(13)+(23) \qquad \alpha_3 = (123)+(132). \tag{2.8}$$

And in the eight elements of the algebra for S_4:

$$\begin{aligned}
&\alpha_1 = e\\
&\alpha_2 = (12)+(14)+(23)+(34)\\
&\alpha_3 = (13)+(24)\\
&\alpha_4 = (123)+(132)+(124)+(142)+(134)+(143)+(234)+(243)\\
&\alpha_5 = (12)(34)+(14)(23)\\
&\alpha_6 = (13)(24)\\
&\alpha_7 = (1234)+(1432)\\
&\alpha_8 = (1342)+(1243)+(1324)+(1423).
\end{aligned} \tag{2.9}$$

By definition, all elements α in (2.8) and (2.9) are essentially invariant under conjugation by elements of the dihedral group:

$$d\alpha_i d^{-1} = k(i)\alpha_i \tag{2.10}$$

$k(i)$ being a rational constant (the index i is not summed), $d \in D_n$.

Furthermore, they are the invariant elements of the algebra A_n with lowest dimensionality, i.e. no invariant linear combinations of permutations $\pi_R \in S_n$ exist with fewer elements π_R.

We expect that physical operators should be invariant under conjugation by elements $d \in D_n$. Furthermore, some of them may be expressed as elements of A_n. In this case those operators should be themselves linear combinations of the elements α. In the next section, we discuss the one-row transfer matrix for a combinational problem, which has the above-mentioned properties.

In order to compute indefinite powers of operators of this type, one would like to rewrite each element of the invariant basis α as linear combination of further elements $P_j \in A_n$ which act as projectors in the algebra:

$$P_i P_j = \delta_{ij} P_i \qquad \text{identity} = \Sigma P_i \qquad bP_i = P_i b = k(b) P_i \tag{2.11}$$

with $k(b)$ being a real constant, for each invariant element $b \in A_n$.

Since the projectors are themslves invariant elements of A_n, their generic forms are

$$P = \sum_i c_i \alpha_i. \tag{2.12}$$

By substituting the expressions (2.12) into (2.11), the coefficients c_i may be found. Yet is more expedient to use just the last of (2.11) by requiring $\alpha_h P = k(h) P$ for a couple of basis elements α_h. This is usually enough to determine all projectors, except those for which $k(h)=0$. In either case the basic tool is the (commutative) multiplication table of the basis elements, which we provide in the appendix.

Then we find the projectors for the algebra A_3:

$$P_1=(\alpha_1-\alpha_2+\alpha_3)/6 \qquad P_2=(\alpha_1+\alpha_2+\alpha_3)/6 \qquad P_3=(2\alpha_1-\alpha_3)/3. \tag{2.13}$$

And for the algebra A_4:

$$\begin{aligned}
P_1&=(\alpha_1-\alpha_2-\alpha_3+\alpha_4+\alpha_5+\alpha_6-\alpha_7-\alpha_8)/24\\
P_2&=(\alpha_1+\alpha_6+\alpha_7-\alpha_3-\alpha_5)/8\\
P_3&=(2\alpha_1-2\alpha_6-\alpha_2+\alpha_8)/8\\
P_4&=[2(\alpha_1+\alpha_5+\alpha_6-\alpha_3-\alpha_7)+(\alpha_2+\alpha_8-\alpha_4)]/24\\
P_5&=[2(\alpha_1+\alpha_3+\alpha_5+\alpha_6+\alpha_7)-(\alpha_2+\alpha_4+\alpha_8)]/24\\
P_6&=(\alpha_1+\alpha_3+\alpha_6-\alpha_5-\alpha_7)/8\\
P_7&=\frac{1}{24}\left(\sum_1^8\alpha_i\right)\\
P_8&=(2\alpha_1-2\alpha_6+\alpha_2-\alpha_8)/8.
\end{aligned} \tag{2.14}$$

The basis elements of A_3 may be written as linear combinations of the projectors:

$$\alpha_1=\sum_1^3 P_i \qquad \alpha_2=-3P_1+3P_2 \qquad \alpha_3=2P_1+2P_2-P_3. \tag{2.15}$$

Similarly for the basis elements of A_4 we obtain.

$$\begin{aligned}
\alpha_1&=\sum_1^8 P_i\\
\alpha_2&=-4P_1-2(P_3-P_4+P_5)+4P_7+2P_8\\
\alpha_3&=-2P_1-2P_2+2(-P_4+P_5)+2(P_6+P_7)\\
\alpha_4&=8P_1-4(P_4+P_5)+8P_7\\
\alpha_5&=2P_1-2P_2+2(P_4+P_5)+2(-P_6+P_7)\\
\alpha_6&=P_1+P_2-P_3+P_4+P_5-P_6+P_7-P_8\\
\alpha_7&=-2P_1+2P_2+2(-P_4+P_5)+2(-P_6+P_7)\\
\alpha_8&=-4P_1-2(-P_3-P_4+P_5)+4P_7-2P_8.
\end{aligned} \tag{2.16}$$

3. Diagonalisation of the transfer matrix on a strip

We study in this section a colouring problem, solved by Baxter for the planar honeycomb lattice, unbounded in every direction, with toroidal boundary conditions. To each bond there corresponds a random variable, or colour, which takes one of three possible values and the problem is the computation of the number of proper colourings (such that three different colours are incident on each vertex). As with Baxter, it is useful to distort the honeycomb cells to produce rows of rectangularly shaped hexagons such that each horizontal row of cells is horizontally translated by half a cell with respect to its two adjacent horizontal rows.

We study this problem for a lattice where each row has a finite number of cells, with cylindrical boundary conditions, all rows have the same number of cells, and the number of rows is unbounded. We consider the transfer matrix for one row with n cells, its indices of rows and columns labelling the possible values of colours on the n vertical bonds above and the n vertical bonds below the horizontal row. That is, the transfer matrix is a square matrix with 3^n rows, where the row index corresponds to a colouring of the vertical bonds above the horizontal line, and the column index corresponds to one of the 3^n colourings of the vertical bonds below the horizontal line; each entry is a non-negative integer corresponding to the number of proper colourings of the horizontal row of cells with fixed external vertical coloured bonds.

Of course the matrix product of transfer matrices corresponding to two adjacent horizontal rows is a square matrix of the same dimension, whose entries are the number of proper colourings for a lattice with two horizontal rows and specified vertical external coloured bonds.

Before evaluating the transfer matrix for one row with three cells and that for one row with four cells, we recall the relation of this colouring problem with the residual entropy of a four-state Potts antiferromagnetic model at zero temperature.

Indeed it is known that a proper colouring of the bonds of this lattice with three colours corresponds to a proper colouring of the faces of the same lattice with four colours [1]. Furthermore it is well known that for planar graphs, regions (or faces) are exchanged into vertices (or sites) by considering the dual graph. Then the number of proper colourings for the bonds of the honeycomb lattice with three colours is equivalent to the number of proper colourings of the sites (with four colours) of the dual lattice (i.e. equilateral triangles with six bonds incident on each vertex), the latter number of colourings being the residual entropy of a four-state antiferromagnetic model on the second lattice.

In order to evaluate the transfer matrix for one row and a given number of cells, we find it convenient to recall the Penrose method for relating the number of proper colourings of a trivalent planar graph to the product of m structure constants of SU(2), m being the number of vertices of the graph [6]. The latter product is usefully evaluated by the general graphic rules given by Cvitanović for the products of structure constants of any simple Lie algebra [7].

The transfer matrix for a horizontal row with three cells may be written

$$T_{(3)} = 4\alpha_1 - 2\alpha_2 + 3\alpha_3 \tag{3.1}$$

with the basis elements α as given in (2.8).

The transfer matrix for a horizontal row with four cells may be written

$$T_{(4)} = 12\alpha_1 - 8\alpha_2 - 6\alpha_3 + 6\alpha_4 + 6\alpha_5 + 2\alpha_6 - 5\alpha_7 - 4\alpha_8. \tag{3.2}$$

By use of (2.15) and (2.16), the transfer matrices are explicitly diagonalised:

$$T_{(3)} = 16P_1 + 4P_2 + P_3 \tag{3.3}$$

$$T_{(4)} = 144P_1 + 4P_2 + 18P_3 + 4P_5 - 4P_6 + 4P_7 + 2P_8. \tag{3.4}$$

For purpose of comparison between the residual entropy per site in a strip of finite width and infinite length (with toroidal boundary conditions) and the same entropy for the unbounded strip in both directions (evaluated by Baxter), just the largest eigenvalue of the transfer matrix is relevant. One may define the residual entropy per site as

$$S = K \log Z \qquad Z = \lim_{N\to\infty} (\lambda^N)^{1/MN} \tag{3.5}$$

where λ is the largest eigenvalue of the transfer matrix (provided the corresponding eigenspace has non-vanishing dimensionality) for one row with c cells, M is the number of sites on a row, $M=4c$, N being the number of horizontal rows. We find $Z_3=(16)^{1/12}=2^{1/3}$ for the strip with three hexagons in each row and $Z_4=(18)^{1/16}\sim 1.197\,99$ for the strip with four hexagons in each row. This differs by less than 1% from the exact value, computed by Baxter for the strip with infinitely many horizontal cells, which is $Z_\infty \sim 1.208\,72$.

Appendix

The products of the three basic elements of the algebra A_3, given in (2.8) are

$$\begin{aligned} &\alpha_1\alpha_i=\alpha_i && (\alpha_2)^2=3\alpha_1+3\alpha_3 \\ &\alpha_2\alpha_3=2\alpha_2 && (\alpha_3)^2=2\alpha_1+\alpha_3. \end{aligned} \tag{A1}$$

The products among the eight basis elements of the algebra A_4, given in (2.9), are $\alpha_1\alpha_i=\alpha_i$

$$\begin{aligned}
&(\alpha_2)^2=4\alpha_1+\alpha_4+2\alpha_5 \qquad \alpha_2\alpha_3=\alpha_4\\
&\alpha_2\alpha_4=2\alpha_2+4\alpha_3+4\alpha_7+2\alpha_8\\
&\alpha_2\alpha_5=\alpha_2+\alpha_8 \qquad \alpha_2\alpha_6=\alpha_8\\
&\alpha_2\alpha_7=\alpha_4 \qquad \alpha_2\alpha_8=\alpha_4+2\alpha_5+4\alpha_6\\
&(\alpha_3)^2=2\alpha_1+2\alpha_6 \qquad \alpha_3\alpha_4=2\alpha_2+2\alpha_8\\
&\alpha_3\alpha_5=2\alpha_7 \qquad \alpha_3\alpha_6=\alpha_3\\
&\alpha_3\alpha_7=2\alpha_5 \qquad \alpha_3\alpha_8=\alpha_4\\
&(\alpha_4)^2=8\alpha_1+4\alpha_4+8\alpha_5+8\alpha_6 \qquad \alpha_4\alpha_5=2\alpha_4\\
&\alpha_4\alpha_6=\alpha_4 \qquad \alpha_4\alpha_7=2\alpha_2+2\alpha_8\\
&\alpha_4\alpha_8=2\alpha_2+4\alpha_3+4\alpha_7+2\alpha_8\\
&(\alpha_5)^2=2\alpha_1+2\alpha_6 \qquad \alpha_5\alpha_6=\alpha_5\\
&\alpha_5\alpha_7=2\alpha_3 \qquad \alpha_5\alpha_8=\alpha_2+\alpha_8\\
&(\alpha_6)^2=\alpha_1 \qquad \alpha_6\alpha_7=\alpha_7\\
&\alpha_6\alpha_8=\alpha_2 \qquad (\alpha_7)^2=2\alpha_1+2\alpha_6\\
&\alpha_7\alpha_8=\alpha_4 \qquad (\alpha_8)^2=4\alpha_1+\alpha_4+2\alpha_5.
\end{aligned} \tag{A2}$$

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